Direct Proof

Generic Outline of Direct Proof: The following proof will be an example of a direct proof. To prove an implication $P \to Q$,

1. Suppose P.
2. Prove Q.

Definitions:

1. A natural number n is **yellow** if there exists a natural number k so that $n = 4k$.
2. A natural number n is **green** if there exists a natural number k so that $n = 4k + 1$.
3. A natural number n is **pink** if there exists a natural number k so that $n = 4k + 2$.
4. A natural number n is **brown** if there exists a natural number k so that $n = 4k + 3$.

Theorem: If a natural number n is yellow, then n^2 is also yellow.

Proof: Suppose that n is a natural number. We will show that if n is yellow, then n^2 is yellow. First, let n be yellow. By the definition of yellow, there exists a natural number k so that $n = 4k$. It follows that

$$\begin{aligned} n^2 &= (4k)^2 \\ &= 16k^2 \\ &= 4(4k^2). \end{aligned}$$

By the definition of yellow, we see that if n is yellow, then n^2 is also yellow. □

Cases

Generic Outline of Cases Proof: The following proof will be an example of a cases proof. To prove a implication $(P \vee Q) \to R$,

1. Prove $P \to R$.
2. Prove $Q \to R$.

Definitions:

1. An integer n is **yellow** if there exists an integer k so that $n = 4k$.
2. An integer n is **green** if there exists an integer k so that $n = 4k + 1$.
3. An integer n is **pink** if there exists an integer k so that $n = 4k + 2$.
4. An integer n is **brown** if there exists an integer k so that $n = 4k + 3$.

Lemma 1: An integer n is yellow if and only if $n \equiv_4 0$.

Proof: Suppose that n is yellow. This means that there is an integer k with $n = 4k$. It follows that $4k = (n - 0)$ and $n \equiv_4 0$. Now suppose that $n \equiv_4 0$. This means that $4k = (n - 0)$ so there exists a $k \in \mathbb{Z}$ so that $4k = n - 0$. Then, $4k = n$; thus n is yellow.

Lemma 2: An integer n is green if and only if $n \equiv_4 1$.

Proof: Suppose that n is green. This means that there is an integer k with $n = 4k + 1$. It follows that $4k = (n - 1)$ and $n \equiv_4 1$. Now suppose that $n \equiv_4 1$. This means that there exists a $k \in \mathbb{Z}$ so that $4k = (n - 1)$, so $4k = n - 1$. Then, $4k + 1 = n$; thus n is green.

Lemma 3: An integer n is pink if and only if $n \equiv_4 2$.

Proof: Suppose that n is pink. This means that there is an integer k with $n = 4k + 2$. It follows that $4k = (n - 2)$ and $n \equiv_4 2$. Now suppose that $n \equiv_4 2$. This means that $4k = (n - 2)$ for some $k \in \mathbb{Z}$, so $4k = n - 2$. Then, $4k + 2 = n$; thus n is pink.

Lemma 4: An integer n is brown if and only if $n \equiv_4 3$.

Proof: Suppose that n is brown. This means that there is an integer k with $n = 4k + 3$. It follows that $4k = (n - 3)$ and $n \equiv_4 3$. Now suppose that $n \equiv_4 3$. This means that $4k = (n - 3)$ for some $k \in \mathbb{Z}$, so $4k = n - 3$. Then, $4k + 3 = n$; thus n is brown.

Theorem: Every integer n is exactly one of yellow, green, pink, or brown.

Proof: Let $n \in \mathbb{Z}$. We can apply the division algorithm to express $n = 4q + r$ where q and r are unique with $0 \leq r < 4$. Since $0 \leq r < 4$, we know that r is either $0, 1, 2,$ or 3. If $r = 0$, then by the definition of equivalence modulo, $n \equiv_4 0$ then by Lemma 1, n is yellow. If $r = 1$, then by the definition of equivalence modulo, $n \equiv_4 1$ then by Lemma 2, n is green. If $r = 2$, then by the definition of equivalence modulo, $n \equiv_4 2$ then by Lemma 3, n is pink. If $r = 3$, then by the definition of equivalence modulo, $n \equiv_4 3$ then by Lemma 4, n is brown. Thus a number is either yellow, green, pink, or brown. Since, q and r are unique, we see that n cannot be more than one of yellow, green, pink, or brown. $\square$

Biconditional

Generic Outline of Biconditional Proof: The following proof will be an example of a biconditional proof. To prove a biconditional $P \leftrightarrow Q$,

1. Prove $P \to Q$.
2. Prove $Q \to P$.

Theorem: Let n be a natural number. Then n is even if and only if n^2 is even.

Proof: Let n be a natural number. We will prove that n is even if and only if n^2 is even. First, we will show that if n is even, then n^2 is even. Then, we will show that if n^2 is even, then n is even. Suppose that n is even. That means that $2|n$. By the definition of divides, there is some natural number k so that $n = 2k$. It follows that

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2). \end{aligned}$$

By the definition of even, we see that if n is even, then n^2 is also even.

Next, we will show that if n^2 is even, then n is even. We will use the contrapositive to show this. The contrapositive states "if n is not even, then n^2 is not even." Suppose that n is not even, that is n is odd. By the definition of odd, there exists some natural number k so that $n = 2k + 1$. It follows that

$$\begin{aligned} (2k+1)^2 &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

Thus we see that if n is odd, then n^2 is odd. Therefore, we see that if n is not even, then n^2 is not even, affirming the contrapositive. Hence, we have proven that n is even if and only if n^2 is even. $\square$

Disjunction

Generic Outline of Disjunction Proof: The following proof will be an example of a disjunction proof. To prove a disjunction $P \vee Q$,

1. Suppose $\neg P$.
2. Prove Q.

Theorem: Suppose that p is a prime number and that $a, b \in \mathbb{N}$ so that $p|ab$. Then, either $p|a$ or $p|b$.

Proof: Let a and b be natural numbers. Suppose that p is a prime number and that $p|ab$. We will prove that either $p|a$ or $p|b$. To do so, now suppose that p does not divide a. We must show that $p|b$. Since p does not divide a, and since p is prime, the greatest common divisor of p and a is 1. By the Euclidean Algorithm, we see that there are $x, y \in \mathbb{Z}$ so that $1 = xp + ya$. Multiplying this by b yields $b = bxp + yab$. Since $p|ab$, by definition of divides, there exists a $k \in \mathbb{N}$ so that $pk = ab$. It follows that

$$\begin{aligned} b &= bxp + ykp \\ &= p(bx + yk) \end{aligned}$$

Since $bx + yk$ is a natural number, we see that $p|b$. Thus, if $p \not| a$ we see that $p|b$. Therefore, $p|a$ or $p|b$. $\square$

Contrapositive

Generic Outline of Contrapositive Proof: The following proof will be an example of a contrapositive proof. To prove an implication $P \to Q$,

1. Suppose $\neg P$.
2. Prove Q.

Definitions:

1. A natural number n is **yellow** if there exists a natural number k so that $n = 4k$.
2. A natural number n is **green** if there exists a natural number k so that $n = 4k + 1$.
3. A natural number n is **pink** if there exists a natural number k so that $n = 4k + 2$.
4. A natural number n is **brown** if there exists a natural number k so that $n = 4k + 3$.

Theorem: Let n be a natural number. If $3n + 1$ is yellow, then n is green.

Proof: Suppose that n is a natural number. We will prove that if $3n + 1$ is yellow, then n is green. To do this, we will use the contrapositive which states, "If n is not green, then $3n+1$ is not yellow." Suppose that n is not green. Since a natural number can be precisely one of yellow, green, pink, or brown, if n is not green, n is either yellow, pink, or brown. First, suppose that n is yellow. By the definition of yellow, there exists a natural number k so that $n = 4k$. It follows that

$$\begin{aligned} 3n + 1 &= 3(4k) + 1 \\ &= 12k + 1 \\ &= 4(3k) + 1 \end{aligned}$$

Thus, by the definition of green, we see that if n is yellow, then $3n+1$ is green and not yellow.

Next, suppose that n is pink. By the definition of pink, there exists a natural number k so that $n = 4k + 2$. It follows that

$$\begin{aligned} 3n + 1 &= 3(4k + 2) + 1 \\ &= 12k + 7 \\ &= 4(3k + 1) + 3 \end{aligned}$$

Thus, by the definition of brown, we see that if n is pink, then $3n+1$ is brown and not yellow.

Next, suppose that n is brown. By the definition of brown, there exists a natural number k so that $n = 4k + 3$. It follows that

$$\begin{aligned} 3n + 1 &= 3(4k + 3) + 1 \\ &= 12k + 10 \\ &= 4(3k) + 2 \end{aligned}$$

Thus, by the definition of pink, we see that if n is brown, then $3n+1$ is pink and not yellow.

Therefore, we see that when n is not green, $3n + 1$ is not yellow affirming the contrapositive. Consequently, we have shown that if $3n + 1$ is yellow, then n is green. $\square$

Contradiction

Generic Outline of Contradiction Proof: The following proof will be an example of a contradiction proof. To prove P, 1. Suppose $\neg P$.
2. Show a contradiction.

Theorem: There are infinitely many prime numbers.

Proof: We will prove by contradiction that there are infinitely many prime numbers. Assume the contrary, that there are a finite number of prime numbers. We will denote all the prime numbers as $p_1, p_2, p_3, \cdots, p_n$ where p_n is the last prime number. Consider the number $q = p_1(p_2)(p_3)\cdots(p_n) + 1$. The number q is either prime or composite. If we divide any of the listed primes p_i into q, there would result a remainder of 1 for each $i = 1, 2, 3, \cdots n$. Thus, we see that q is not composite. Since a number is either prime or composite, q must be prime yet not included in the list $p_1, p_2, p_3, \cdots, p_n$. This is a contradiction, because the list of all prime numbers does not include the prime number q. Therefore, we see that there are infinitely many prime numbers. □

Induction

Generic Outline of Proof by Induction: The following proof will be an example of an induction proof. To prove that $P(n)$ is true for all integers $n \geq m$,

1. Prove $P(m)$.
2. Let $k \in \mathbb{Z}$ with $m \leq k$.
3. Suppose $P(k)$.
4. Prove $P(k+1)$.

Theorem: For any natural number n, $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} < 2$

Proof: Let $P(n)$ be the open statement "$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} < 2$." We will prove through mathematical induction that $P(n)$ is true for all natural numbers. First, note that $P(0)$ is true, because $\frac{1}{2^0} = 1 < 2$. Next, suppose that $P(k)$ is true for some natural number k. That means $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^k} < 2$. Observe that multiplying by $\frac{1}{2}$ yields $\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^k}+\frac{1}{2^{k+1}} < 1$. Then adding 1 gives $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} < 2$

Thus, we see that $P(0)$ is true and that $P(k)$ implies $P(k+1)$ for all $k \in \mathbb{N}$. By mathematical induction, we can conclude that $P(n)$ is true for all natural numbers. □

Subset

Generic Outline of Subset Proof: The following proof will be an example of a subset proof. To prove that a set A is a subset of a set B,

1. Let $a \in A$.
2. Prove $a \in B$.

Theorem: Suppose that set A is defined by $A = \{n \in \mathbb{N} : 6|n\}$, and suppose set B is defined by $B = \{n \in \mathbb{N} : 2|n\}$. Then, $A \subseteq B$.

Proof: Suppose that set A is defined by $A = \{n \in \mathbb{N} : 6|n\}$, and suppose set B is defined by $B = \{n \in \mathbb{N} : 2|n\}$. We will prove that $A \subseteq B$. Let $x \in A$ be arbitrary. By the definition of A, $6|x$. By the definition of divides, there exists some natural number k so that $x = 6k$. It follows that $x = 2(3k)$. Thus, we see that $2|x$. Therefore x is also in B. Since an arbitrary element of A is also an element in B, we see that $A \subseteq B$. $\square$

Set Equality

Generic Outline of Proof of Set Equality: The following proof will be an example of a set equality proof. To prove that a set A equals a set B,

1. Prove $A \subseteq B$.
2. Prove $B \subseteq A$.

Theorem: Let A, B, and C be sets. Then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof: Let A, B, C be sets. We will prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. First, we will show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Then we will show that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Suppose that $x \in A \cap (B \cup C)$. By the definition of intersection, $x \in A$. and $x \in (B \cup C)$. By the definition of union, $x \in B$ or $x \in C$. If $x \in B$, then $x \in A$ and $x \in B$, so $x \in (A \cap B)$. If $x \in C$, then $x \in A$ and $x \in C$, so $x \in (A \cap C)$. We see that either $x \in (A \cap B)$ or $x \in (A \cap C)$. By the definition of union, this means that $x \in (A \cap B) \cup (A \cap C)$. Therefore, we can conclude that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Next let $x \in (A \cap B) \cup (A \cap C)$. By the definition of union, $x \in (A \cap B)$ or $x \in (A \cap C)$. According to the definition of intersection, either $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$. By the distributive law, $x \in A$ and $x \in B$ or $x \in C$. By the definition of union, $x \in A$, and $x \in (B \cup C)$. By the definition of intersection, $x \in A \cap (B \cup C)$. Thus, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Since $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$, then we can conclude that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

Existential

Generic Outline of Proof of an Existential Statement: The following is an example of an existential statement proof. To prove an existential statement, exhibit an instance when the statement is true.

Theorem: There exists a natural number k so that $k^k = k + k$.

Proof: We will prove that there exists a natural number k so that$k^k = k + k$. Note that $2^2 = 4$ and $2 + 2 = 4$. Thus 2 is an example of a natural number k that satisfies $k^k = k + k$.

Universal

Generic Outline of Proof of a Universal Statement: The following is an example of a universal statement proof. To show that a statement is true for all

Theorem: For all odd natural numbers n, $2n^2 + 44n - 23$ is odd.

Proof: Let n be an odd natural number. We will prove that $2n^2 + 44n - 23$ is odd. Note that

$$2n^2 + 44n - 23 = 2(n^2 + 22n - 12) + 1.$$

Thus, we see by the definition of odd that, $2n^2 + 44n - 23$ is odd. □

Inverse Function

Generic Outline of Proof that two functions are inverses: The following will be an example of a proof showing that two functions are inverses. To prove that $g : B \to A$ is the inverse of $f : A \to B$,

1. Let $a \in A$
2. Prove $g(f(a)) = a$
3. Let $b \in B$
4. Prove $f(g(b)) = b$

Theorem: The functions $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 4x - 16$ and $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \frac{1}{4}x + 4$ are inverses.

Proof: We will show that the functions $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 4x - 16$ and $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \frac{1}{4}x + 4$ are inverses. Let $a \in \mathbb{R}$ be arbitrary. It follows that

$$\begin{aligned} g(f(a)) &= \frac{1}{4}(f(a)) + 4 \\ &= \frac{1}{4}(4a - 16) + 4 \\ &= a - 4 + 4 \\ &= a. \end{aligned}$$

Thus we see that $g(f(a)) = a$ for any $a \in \mathbb{R}$.

Next, let $b \in \mathbb{R}$ be arbitrary. It follows that

$$\begin{aligned} f(g(b)) &= 4(f(b)) - 16 \\ &= 4(\frac{1}{4}b + 4) - 16 \\ &= b + 16 - 16 \\ &= b. \end{aligned}$$

Thus we see that $f(g(b)) = b$ for any $b \in \mathbb{R}$. Therefore, we can conclude that f and g are inverses. □

Injectivity

Generic Outline of Proof of Injectivity: The following will be an example of a proof showing that a function is injective. To prove a function's injectivity,

1. Suppose $f(x) = f(y)$.
2. Prove $x = y$.

Theorem: Suppose that $f : [0, 1] \to \mathbb{R}$ given by $f(x) = x^2$ is a function. Then f is injective.

Proof: Suppose that $f : [0, 1] \to \mathbb{R}$ given by $f(x) = x^2$ is a function. We will prove that f is injective. Let $x, y \in [0, 1]$ and suppose that $f(x) = f(y)$. Then $x^2 = y^2$. Taking the square root yields $\pm x = \pm y$. Since $x, y \in [0, 1]$, x and y are never negative; thus $x = y$. Therefore, we see that if $f(x) = f(y)$, then $x = y$. Consequently, f is injective. $\square$

non-Injectivity

Generic Outline of Proof of non-Injectivity: The following will be an example of a proof showing that a function is not injective. To prove a function $f : A \to B$ is not injective,

1. Give an example of $x, y \in A$ so that $x \neq y$ but $f(x) = f(y)$.

Theorem: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not injective.

Proof: We will show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not injective. Note that 1 and -1 are real numbers and that $f(-1) = 1 = f(1)$. Therefore, we see that f is not injective. $\square$

Surjectivity

Generic Outline of Proof of Surjectivity: The following will be a proof showing that a function is surjective. To prove that a function $f : A \to B$ is surjective,

1. Let $b \in B$ be arbitrary.
2. Give an example of an element $a \in A$ so that $f(a) = b$.

Theorem: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{2}x + 4$ is surjective.

Proof: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \frac{1}{2}x + 4$. We will show that f is surjective. Let $b \in \mathbb{R}$ be arbitrary and let $a = 2b - 8$. Then, $a \in \mathbb{R}$. It follows that

$$\begin{aligned} f(a) &= \frac{1}{2}(a) + 4 \\ &= \frac{1}{2}(2b - 8) + 4 \\ &= b - 4 + 4 \\ &= b. \end{aligned}$$

Thus, there is an $a \in \mathbb{R}$ so that $f(a) = b$. Therefore, f is surjective. $\square$

non-Surjectivity

Generic Outline of Proof of non-Surjectivity: The following proof is an example of a proof showing that a function is not surjective. To show that a function $f : A \to B$ is not surjective,

1. Give an example of an element $b \in B$ so that there can be no $a \in A$ with $f(a) = b$.

Theorem: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 4x^2 + 9$ is not surjective.

Proof: We will prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 4x^2 + 9$ is not surjective. Note that for all $x \in \mathbb{R}$, $f(x) = 4x^2 + 9 \geq 9$. Therefore, there can be no $x \in \mathbb{R}$ with $f(x) = 4$. Thus, f is not surjective. $\square$

Bijectivity

Generic Outline of Proof of Bijectivity: The following will be an example of a proof showing that a function is bijective. To prove that a function $f : A \to B$ is bijective,

1. Prove that f is injective.
2. Prove that f is surjective.

Theorem: Suppose that D is the set of all odd integers. The function $f : \mathbb{Z} \to D$ given by $f(x) = 2x + 1$ is bijective.

Proof: Suppose that D is the set of all odd integers. We will prove that the function $f : \mathbb{Z} \to D$ given by $f(x) = 2x + 1$ is bijective. We will first show that f is injective, then we will show that f is surjective. Let $x, y \in \mathbb{Z}$ and suppose that $f(x) = f(y)$. That is $2x + 1 = 2y + 1$. Subtracting 1 and multiplying by $\frac{1}{2}$ yields $x = y$. Therefore, if $f(x) = f(y)$ then $x = y$. Thus, f is injective.

Now, we will show that f is surjective. Let $b \in D$ and let $a = \frac{1}{2}b - \frac{1}{2}$. Then $a \in \mathbb{Z}$. It follows that

$$\begin{aligned} f(a) &= 2(a) + 1 \\ &= 2(\frac{1}{2}b - \frac{1}{2}) + 1 \\ &= b - 1 + 1 \\ &= b. \end{aligned}$$

Thus, there is an $a \in \mathbb{Z}$ so that $f(a) = b$. Therefore, f is surjective. Since f is injective and surjective, we can conclude that f is bijective. $\square$

Equivalence Relation

Generic Outline of Equivalence Relation Proof: The following proof will be an example of a proof showing that a relation is an equivalence relation. To prove that a relation R is an equivalence relation,

1. Show that R is reflexive.
2. Show that R is symmetric.
3. Show that R is transitive.

Theorem: Suppose R is a relation on $\mathbb{R}$ defined by xRy if and only if $2\cos(x) = 2\cos(y)$. Then, R is an equivalence equation.

Proof: Suppose R is a relation on $\mathbb{R}$ defined by xRy if and only if $2\cos(x) = 2\cos(y)$. We will prove that R is an equivalence equation. First, let $x \in \mathbb{R}$. Since $2\cos(x) = 2\cos(x)$ is true, xRx. Thus R is reflexive.

Next, let $x, y \in \mathbb{R}$ and suppose that xRy. That means $2\cos(x) = 2\cos(y)$. So, it is also true that $2\cos(y) = 2\cos(x)$. Since xRy implies yRx, R is symmetric.

Finally, let $x, y, z \in \mathbb{R}$ and suppose that xRy and yRz. That means $2\cos(x) = 2\cos(y)$ and $2\cos(y) = 2\cos(z)$. Therefore, $2\cos(x) = 2\cos(z)$. Thus R is transitive. Since R is reflexive, symmetric, and transitive, R we see that R is an equivalence relation on $\mathbb{R}$. $\square$

Cardinality

Generic Outline of Cardinality Proof: The following will be an example of a cardinality proof. To prove that two sets A and B have the same cardinality, exhibit a bijection from A to B.

Theorem: Suppose that D is the set of all odd natural numbers. The set D has the same cardinality as the set of all natural numbers.

Proof: Suppose that D is the set of all odd natural numbers. D has the same cardinality as the set of all natural numbers, because the function $f : \mathbb{N} \to D$ given by $f(x) = 2x + 1$ is a bijection. $\square$

Cantor-Schroeder-Bernstein

Generic Outline of Cantor-Schroeder-Bernstein Proof: The following will be an example of a proof using the Cantor-Schroeder-Bernstein theorem. To prove that two sets A and B have the same cardinality,

1. Exhibit an injection from A to B.
2. Exhibit an injection from B to A.

Theorem: The cardinality of the interval $[1,3]$ is the same as the cardinality of the interval $(2,10)$.

Proof: We will prove that the cardinality of the interval $[1,3]$ is the same as the cardinality of the interval $(2,10)$. Note that the function $f:[1,3]\to(2,10)$ given by $f(x)=x^2+1$ is injective. Also note that the function $g:(2,10)\to[1,3]$ given by $g(x)=(x+1)^{\frac{1}{2}}$ is injective. Since there is an injection from $[1,3]$ to $(2,10)$ and an injection from $(2,10)$ to $[1,3]$, we can conclude that $|[1,3]|=|(2,10)|$. $\square$