Direct Proof

Generic Outline of Direct Proof:
To prove an implication $P \to Q$,

1. Suppose P.
2. Prove Q.

Theorem: For any natural numbers l, m, and n, if $n \leq m$ then $nl \leq ml$

Proof: Suppose l, m, n are natural numbers and $n \leq m$. We will show $nl \leq ml$. By the definition of order, there is a natural number k so that $n + k = m$. It follows that $nl + kl = ml$ so $nl \leq ml$. Therefore, for any natural numbers l, m, n, if $n \leq m$ then $nl \leq ml$. $\square$

Biconditional

Generic Outline of Biconditional Proof:
To prove a biconditional $P \leftrightarrow Q$,

1. Prove $P \to Q$.
2. Prove $Q \to P$.

Theorem: An integer n is even if and only if $n + 2$ is even.

Proof: Suppose that $n \in \mathbb{Z}$. We will show that n is even if and only if $n + 2$ is even. First, we suppose that n is even. By the definition of even, $2|n$. By the definition of divisibility, there is a $k \in \mathbb{Z}$ so that $n = 2k$. Adding two to both sides gives $n + 2 = 2k + 2 = 2(k + 1)$. Hence, $2|(n + 2)$. Therefore, $n + 2$ is even.

Next, suppose $n + 2$ is even. By the definition of even, $2|(n + 2)$. By the definition of divisibility, there is a natural number l so that $n + 2 = 2l$. Subtracting two from both sides gives $n = 2l - 2 = 2(l - 1) + 2$. Hence, $2|(n + 2)$. Thus, n is even. Therefore, n is even when $n + 2$ is even.

We have shown that an integer n is even if and only if $n + 2$ is even. $\square$

Disjunction

Generic Outline of Disjunction Proof:
To prove a disjunction $P \vee Q$,

1. Suppose $\neg P$.
2. Prove Q.

Theorem: If n is a natural number, either n is even or $3n$ is odd.

Proof: Suppose n is a natural number. We will prove that either n is even or $3n$ is odd. To do so, suppose n is not even. Since every number is either even or odd and not both, n is odd. We must show $3n$ is odd. By definition of odd, there is a natural number k so that $n = 2k + 1$. It follows that

$$\begin{aligned} 3n &= 3(2k+1) \\ &= 6k+3 \\ &= 6k+2+1 \\ &= 2(3k+1)+1. \end{aligned}$$

Thus, $3n$ is odd.

We have shown if n is not even, then $3n$ is odd. Therefore, if n is a natural number, either n is even, or $3n$ is odd. $\square$

Contrapositive

Generic Outline of Contrapositive Proof:
To prove $P \to Q$,

1. Sppose $\neg Q$.
2. Prove $\neg P$.

Theorem: For any number n, if n^3 is even, then n is even.

Proof: Let n be a natural number. We will use the contrapositive to prove that if n^3 is even, then n is even. The contrapositive is, "If n is not even, then n^3 is not even." Suppose n is not even. Since every number is either even or odd, n is odd. Since n is odd, there is some natural number k so that $n = 2k + 1$. It follows that

$$\begin{aligned} n^3 &= (2k+1)^3 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2(4k^3 + 6k^2 + 3k) + 1. \end{aligned}$$

Hence, n^3 is odd, and not even.

We have proven that if n is not even, then n^3 is not even. This is the contrapositive of the theorem. $\square$

Contradiction

Generic Outline of Contradiction Proof:
To prove P,

1. Suppose $\neg P$.

2. Prove a contradiction.

3. Conclude P.

Theorem: If n is a natural number and n^3 is odd, then n is odd.

Proof: Suppose that n is a natural number and than n^3 is odd. We will use contradiction to prove that n is odd. Suppose n is not odd. Then n is even and there is a natural number k so that $n = 2k$. It follows that

$$\begin{aligned} n^3 &= (2k)^3 \\ &= 8k^2 \\ &= 2(4k^3). \end{aligned}$$

so n^3 is even. But then n^3 is both odd and even. This contradicts the theorem stating every natural number is even or odd and not both, so the assumption that n is not odd must be false. Therefore, it has to be the case that n is odd. $\square$

Induction

Generic Outline of Proof by Induction:
To prove that $P(n)$ is true for all integers $n \geq m$,

1. Prove $P(m)$.
2. Let $k \in \mathbb{Z}$ with $m \leq k$.
3. Suppose $P(k)$.
4. Prove $P(k+1)$.

Theorem: For any natural number n, $5^{2n+1}+1$ is divisible by 6.

Proof: Let $P(n)$ be the open statement "$5^{2n+1}+1$ is divisible by 6". We will use induction to show that $P(n)$ is true for all natural numbers n. First, note that $5^{2\cdot 0+1}+1$ is divisible by 6, so $P(0)$ is true. Next, suppose $P(k)$ is true for some natural number k. That is, we are assumming $5^{2k+1}+1$ is divisible by 6. This means there is a natural number l so that $5^{2k+1}+1=6l$. Observe

$$\begin{aligned}
5^{2(k+1)+1}+1 &= 5^{2k+3}+1 \\
&= 5^{2k+1}5^2+1 \\
&= 5^{2k+1}\cdot 25+1 \\
&= 5^{2k+1}(4\cdot 6+1)+1 \\
&= 5^{2k+1}\cdot 4\cdot 6+5^{2k+1}+1 \\
&= 5^{2k+1}\cdot 4\cdot 6+6l \\
&= 6(5^{2k+1}\cdot 4+l)
\end{aligned}$$

From the definition of divisibilty, we see that 6 divides $5^{2(k+1)+1}+1$, so $P(k+1)$ is true. That is, if $P(k)$ is true, so is $P(k+1)$.

We have established that $P(0)$ is true and that $P(k)$ implies $P(k+1)$ for all natural numbers k. By mathematical induction, we can conclude that $P(n)$ is true for all natural numbers n. $\square$

Subset

Generic Outline of Subset Proof:
To prove that a set A is a subset of a set B,

1. Let $a \in A$.
2. Prove $a \in B$.

Theorem: The set $A = \{x \in R : x^2 - 4 = 0\}$ is a subset of the set $B = \{x \in R : x^4 - 16 = 0\}$.

Proof: Suppose $x \in A$ is arbitrary. This means that $x^2 - 4 = 0$, so

$$x^4 - 16 = (x^2 - 4)(x^2 + 4) = 0 \cdot (x^2 + 4) = 0.$$

Hence, $x \in B$. Thus every element of A is an element of B. $\square$

Set Equality

Generic Outline of Proof of Set Equality:
To prove a set A equals a set B,

1. Prove $A \subseteq B$.

2. Prove $B \subseteq A$.

Theorem: If A and B are sets then $A - (B \cup C) = (A - B) \cap (A - C)$.

Proof: Let A, B, and C be sets. We will prove that

$$A - (B \cup C) = (A - B) \cap (A - C).$$

First, suppose that $x \in A - (B \cup C)$. From the definition of difference, this means that $x \in A$ and $x \notin (B \cup C)$. From the definition of union and DeMorgan's Law, $x \notin B$ and $x \notin C$. Since $x \notin B$, then $x \in A$ and $x \notin B$, so $x \in (A - B)$. Since $x \notin C$, then $x \in A$ and $x \notin C$, so $x \in (A - C)$. We see that $x \in (A - B)$ and $x \in (A - C)$. This means that $x \in (A - B) \cap (A - C)$. Thus,

$$A - (B \cup C) \subseteq (A - B) \cap (A - C).$$

Now, let $x \in (A-B) \cap (A-C)$. This means that $x \in A$ and $x \notin B$ and $x \in A$ and $x \notin C$. Thus using DeMorgan's Law, $x \in A$ and $x \notin (B \cup C)$. This means $x \in A - (B \cup C)$. Thus,

$$(A - B) \cap (A - C) \subseteq A - (B \cup C).$$

Since

$$A - (B \cup C) \subseteq (A - B) \cap (A - C)$$

and

$$(A - B) \cap (A - C) \subseteq A - (B \cap C)$$

we know

$$A - (B \cup C) = (A - B) \cap (A - C).$$

$\square$

Existential

Generic Outline of Proof of an Existential Statement:
To prove $(\exists x \in A)P(x)$

1. Exhibit some $x \in A$ so $P(x)$ is true.

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2+5$. There is an $x \in \mathbb{N}$ so that $f(x)$ is even.

Proof: Let $x \in \mathbb{N}$. Note that $f(3) = 14$. $\square$

Universal

Generic Outline of Proof of a Universal Statement:
To prove $(\forall x \in A)P(x))$

1. Let $x \in A$.
2. Show $P(x)$.

Theorem: For all natural numbers n, $2n+3$ is odd.

Proof: Suppose that n is any natural number. We will prove that $2n+3$ is odd. Note that

$$\begin{aligned} 2n+3 &= 2n+2+1 \\ &= 2(n+1)+1. \end{aligned}$$

Therefore, $2n+3$ is odd for all natural numbers n. $\square$

Cases

Generic Outline of Proof by Cases:
To prove $(P \vee Q) \to R$,

1. Prove $P \to R$.
2. Prove $Q \to R$.

Theorem: If n is a natural number, then $n^3 + 3n$ is even.

Proof: Suppose that n is a natural number. We will prove that $n^3 + 3n$ is even. There are two cases - either n is even or n is odd. Suppose first that n is even. Then there is a natural number k so that $n = 2k$. It follows that

$$\begin{aligned} n^3 + 3n &= (2k)^3 + 3(2k) \\ &= 8k^3 + 6k \\ &= 2(4k^3 + 3k). \end{aligned}$$

Thus $2|(n^3 + 3n)$. Therefore, if n is even, then $n^3 + 3n$ is even.

Next, suppose that n is odd. Then there is a natural number k so that $n = 2k + 1$. It follows that

$$\begin{aligned} n^3 + 3n &= (2k + 1)^3 + 3(2k + 1) \\ &= (2k + 1)(4lk^2 + 4k + 1) + 6l + 3 \\ &= 8k^3 + 12k^2 + 6k + 4 \\ &= 2(4k^3 + 6k^2 + 3k + 2). \end{aligned}$$

Thus, $2|(n^3 + 3n)$. Therefore, if n is odd, then $n^3 + 3n$ is also even.

We have proven that if n is either even or odd, then $n^3 + 3n$ is even. Since every natural number is either even or odd, $n^3 + 3n$ is even for all natual numbers n. $\square$

Inverse Function

Generic Outline of Proof that two functions are inverses:
To prove that $f : A \to B$ and $g : B \to A$ are inverses,

1. Let $a \in A$.
2. Prove $g(f(a)) = a$.
3. Let $b \in B$.
4. Prove that $f(g(b)) = b$.

Theorem: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \dfrac{x-2}{3}$ is the inverse of the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = 3x + 2$.

Proof: Let $a \in \mathbb{R}$. We calculate $f(g(a))$.

$$\begin{aligned} g(f(a)) &= f(3a+2) \\ &= \frac{(3a+2)-2}{3} \\ &= \frac{3a}{3} \\ &= a. \end{aligned}$$

Thus, $f(g(a)) = a$.

Next, we let $b \in \mathbb{R}$. We calculate $g(f(b))$.

$$\begin{aligned} g(f(b)) &= g\left(\frac{b-2}{3}\right) \\ &= 3(\frac{b-2}{3}) + 1 \\ &= (b-1) + 1 \\ &= b. \end{aligned}$$

Thus, $g(f(b)) = b$.

Since $f(g(a)) = a$ and $g(f(b)) = b$ for all a and b in $\mathbb{R}$, f is the inverse of g. $\square$

Injectivity

Generic Outline of Proof of Injectivity:
To prove that a function $f : A \to B$ is injective,

1. Let $x, y \in A$.
2. Suppose that $f(x) = f(y)$.
3. Prove that $x = y$.

Theorem: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x - 1$ is injective.

Proof: Let $x, y \in \mathbb{R}$ and assume that $f(x) = f(y)$. We will show that $x = y$. Since $f(x) = f(y)$, it follows that $x - 1 = y - 1$. Adding 1 to both sides of this equation yields $x = y$. We have shown that if $f(x) = f(y)$ then $x = y$. It follows that f is injective. $\square$

non-Injectivity

Generic Outline of Proof of non-Injectivity:
To show a function $T : A \to B$ is not injective, exhibit two elements x and y in A so that $x \neq y$ but $f(x) = f(y)$.

Theorem: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 2$ is not injective.

Proof: Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + 2$. Note that $f(-1) = 3 = f(1)$. Since $-1 \neq 1$, f is not injective. $\square$

Surjectivity

Generic Outline of Proof of Surjectivity:
To prove that a function $f : A \to B$ is surjective,

1. Let $b \in B$.
2. Exhibit an $a \in A$ with $f(a) = b$.

Theorem: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 4x^7 + 3$ is surjective.

Proof: We prove that the function $f(x) = 4x^7 + 3$ from $\mathbb{R}$ to $\mathbb{R}$ is surjective. Let $b \in \mathbb{R}$, and let $a = \left(\frac{b-3}{4}\right)^{\frac{1}{7}}$. Note that $a \in \mathbb{R}$ and

$$\begin{aligned} f(a) &= 4\left(\left(\frac{b-3}{4}\right)^{\frac{1}{7}}\right)^7 + 3 \\ &= 4\left(\frac{b-3}{4}\right) + 3 \\ &= (b-3) + 3 \\ &= b. \end{aligned}$$

Thus, for every $b \in \mathbb{R}$, there is an $a \in \mathbb{R}$ with $f(a) = b$. The function f is surjective. $\square$

non-Surjectivity

Generic Outline of Proof of non-Surjectivity:
To show that a transformation $T : A \to B$ is not surjective, exhibit an element $b \in B$ so that there can be no $a \in A$ with $T(a) = b$.

Theorem: The function $f(x) = x^4 + 2$ from $\mathbb{R}$ to $\mathbb{R}$ is not surjective.

Proof: Notice that for all $x \in \mathbb{R}$, $f(x) = x^4 + 2 \geq 2$. Therefore, there can be no $x \in \mathbb{R}$ with $f(x) = 0$. Hence f is not surjective. $\square$

Bijectivity

Generic Outline of Proof of Bijectivity:
To prove that a function $f : A \to B$ is bijective,

1. Prove that f is injective.

2. Prove that f is surjective.

Theorem: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 + 3$ is bijective.

Proof: Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^3 + 3$. We will prove f is bijective. By the definition of bijective, f is a bijection if and only if f is both injective and surjective. First, we will prove f is injective. Let $x, y \in \mathbb{R}$, and suppose $f(x) = f(y)$. Then $x^3 + 3 = y^3 + 3$. Subtracting 3 and taking the cube root of both sides of the equation gives $x = y$. Thus, f is injective.

Next, we will prove f is surjective. Let $b \in R$. Let $a = (b-3)^{\frac{1}{3}}$. Then, $a \in R$ and

$$\begin{aligned}
f(a) &= a^3 + 3 \\
&= ((b-3)^{\frac{1}{3}} + 3) \\
&= (b-3) + 3 \\
&= b.
\end{aligned}$$

Thus, f is surjective.

We have proven that f is both injective and surjective. Hence, f is bijective. $\square$

Equivalence Relation

Generic Outline of Equivalence Relation Proof:
To prove that a binary relation$R$ on a set A is an equivalence relation,

1. Prove that R is reflexive.

2. Prove that R is symmetric.

3. Prove that R is transitive.

Theorem: R is the relation on $\mathbb{R}$ defined by xRy if $x^3 = y^3$.

Proof: Let R be the relation on $\mathbb{R}$ efined by xRy if $x^3 = y^3$. We will prove that R is an equivalence relation on $\mathbb{R}$. First, we will show R is reflexive. Let $x \in \mathbb{R}$. Since $x = x$, it follows that $x^3 = x^3$. Hence xRx for all $x \in \mathbb{R}$. Therefore, R is reflexive.

Next, we will show R is symmetric. Let $x, y \in \mathbb{R}$ be arbitrary and suppose xRy. This means $x^3 = y^3$ then $y^3 = x^3$ so yRx. Since xRy implies yRx, R is symmetric.

Finally, we will show R is transitive. Let $x, y, z \in \mathbb{R}$ be arbitrary and suppose xRy and yRz. Then $x^3 = y^3$ and $y^3 = z^3$, so $x^3 = z^3$. Hence, xRz. Therefore R is transitive.

We have shown R is reflexive, symmetric, and transitive. Therefore, we have proven R is an equivalence relation on $\mathbb{R}$. $\square$

Cardinality

Generic Outline of Cardinality Proof:
To prove that two sets A and B have the same cardinality, exhibit a bijection from A to B.

Theorem: If A is the set [0,1] and B is the set [46,52], then $|A| = |B|$.

Proof: Suppose A is the set [0,1] and B is the set [46,52]. We will prove that $|A| = |B|$. Let $f : A \to B$ be the function given by $f(x) = 6x + 46$. We will prove f is bijective. To prove f is bijective, we will show f is both injective and surjective.

First, we will show f is injective. Let $x, y \in \mathbb{R}$ and assume $f(x) = f(y)$. It follows that $6x + 46 = 6y + 46$. Subtracting 46 from both sides of the equation gives $6x = 6y$. Dividing 6 from both sides yields $x = y$. Thus, f is injective.

Next, we will show f is surjective. Let $y \in B$ and let $x = \dfrac{y-46}{6}$. Note that $x \in A$ and

$$\begin{aligned} f(x) &= 6\left(\frac{y-46}{6}\right) + 46 \\ &= (y-46) + 46 \\ &= y. \end{aligned}$$

Thus, f is surjective.

Therefore, since f is both injective and surjective, we can conclude f is bijective. Since f is bijective, $|A| = |B|$. $\square$

Cantor-Schroeder-Bernstein

Generic Outline of Cantor-Schroeder-Bernstein Proof:
To prove two sets A and B have the same cardinality,

1. Exhibit an injection from A to B.

2. Exhibit an injection from B to A.

Theorem: If A is the set (0,1) and B is the set [7,10], then $|A| = |B|$.

Proof: Suppose A is the set (0,1) and B is the set [7,10]. We will show $|A| = |B|$. Notice that the $f : (0, 1) \to [7, 10]$ given by $f(x) = 3x + 7$ is injective. Also note that $g : [7, 10] \to (0, 1)$ given by $g(x) = \dfrac{x-7}{3}$ is injective. Therefore, we have shown with the Cantor-Schroeder-Bernstein theorem that $|(0, 1)| = |[7, 10]|$. $\square$